Integrálok

Motiváció: Kör területe
Tudjuk, hogy a kör kerülete $2\pi r$. A terület a következő lesz, ha a sugár R:

$$\int 2\pi r dr = 2\pi \int r dr = 2\pi \frac{r^2}{2} = r^2\pi$$

$$\int_0^8 x^5 + 6x^3 + 145x dx = \left[\frac{x^6}{6} + \frac{6x^4}{4} + \frac{145x^2}{2}\right]_0^8 = \frac{2^{18}}{6} + \frac{6 \cdot 2^{12}}{4} + \frac{145 \cdot 64}{2} = 54474 + \frac{2}{3}$$

$$\int_1^2 x^2 \cdot e^{x^3} dx$$

$$f(g(x)) \cdot g'(x)$$

$$f = e^x \quad g = x^3 \quad g' = 3x^2$$

$$F(x) = e^x$$

$$F'(x) = f(x)$$

$$\frac{1}{3}\int_1^2 e^{x^3} \cdot 3x^2 dx = \left[\frac{e^{x^3}}{3}\right]_1^2 = \frac{e^{2^3} - e^{1^3}}{3} = \frac{e^8 - e}{3}$$

Parciális Integrálás $\int f(x) \cdot g'(x) = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$

$$\int_{-\pi/2}^{\pi/2} e^x \cdot \cos(x) dx$$

$f = e^x$	$g' = \cos(x)$
$f' = e^x$	$g = \sin(x)$

$$[e^x \cdot \sin(x)]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} e^x \cdot \sin(x) dx$$

$f = e^x$	$g' = \sin(x)$
$f' = e^x$	$g = -\cos(x)$

$$\Big[e^x \cdot \sin(x)\Big]_{-\pi/2}^{\pi/2} - \Big[\big[e^x \cdot (-\cos(x))\big]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} e^x \cdot (-\cos(x))dx\Big]$$

$$\Big[e^x \cdot (\sin(x) + \cos(x))\Big]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} e^x \cos(x)dx$$

$$2 \cdot \int_{-\pi/2}^{\pi/2} e^x \cos(x)dx = \Big[e^x \cdot (\sin(x) + \cos(x))\Big]_{-\pi/2}^{\pi/2}$$

$$\frac{1}{2} \cdot \Big[e^{\pi/2} \cdot [\sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2})] - e^{-\pi/2} \cdot [\sin(-\frac{\pi}{2}) + \cos(-\frac{\pi}{2})]\Big]$$

$$\frac{1}{2} \cdot \Big[e^{\pi/2} + e^{-\pi/2}\Big]$$

Általánosan nem igaz, de most ennél ez pont $\cosh(\frac{\pi}{2}) \approx 2.509$

Táblázatos parciális integrálás trükk:

$$\int x^4 e^x dx$$

f	g'
x^4	e^x
$4x^3$	e^x
$12x^2$	e^x
$24x$	e^x
24	e^x
0	e^x

$$\int x^4 e^x dx = x^4 \cdot e^x - \left[4x^3 \cdot e^x - \left(12x^2 \cdot e^x - \left[24x \cdot e^x - 24e^x\right]\right)\right] =$$

$$(x^4 - 4x^3 + 12x^2 - 24x + 24)e^x$$

Linearizáló formulák:

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \qquad \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

(Ebből könnyen ki is lehet számolni: $\cos(2x) = \cos^2(x) - \sin^2(x)$)

$$\int_0^{2\pi} \cos^6(x)dx$$

$$\int_0^{2\pi} \cos^2(x)$$

$$\int_0^{2\pi} \frac{1+\cos(2x)}{2}dx$$

$$\int_0^{2\pi} \frac{1}{2}+\frac{\cos(2x)}{2}dx$$

$$\left[\frac{x}{2}+\frac{\sin(2x)}{4}\right]_0^{2\pi}$$

$$\left[\frac{2\pi}{2}+\frac{\sin(4\pi)}{4}\right]-\left[\frac{0}{2}+\frac{\sin(2\cdot 0)}{4}\right]=\pi$$

$$\int \frac{dx}{1-x^2}$$

$$\int \frac{1}{(1-x)(1+x)}dx$$

$$\frac{1}{(1-x)(1+x)}=\frac{A}{1-x}+\frac{B}{1+x}=$$

$$\frac{A(1+x)+B(1-x)}{(1-x)(1+x)}$$

$$\begin{bmatrix} A-B \\ A+B \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A=\frac{1}{2} \quad B=\frac{1}{2}$$

$$\int \frac{\frac{1}{2}}{1-x}+\frac{\frac{1}{2}}{1+x}dx=\frac{1}{2}(\ln|1+x|-\ln|1-x|)$$

$$\int_0^a \sqrt{a^2-x^2}dx$$

Először kiszámoljuk a határozatlan integrált:

$$\int \sqrt{a^2 - x^2} dx =$$

$$x = a\sin(t) \qquad dx = a\cos(t)dt$$

$$\int \sqrt{a^2 - (a\sin(t))^2} \cdot a\cos(t)dt =$$

$$a^2 \int \sqrt{1 - \sin^2(t)}\cos(t)dt =$$

$$a^2 \int \cos^2(t)dt =$$

$$a^2 \int \frac{1 + \cos(2t)}{2} dt =$$

$$\frac{a^2}{2} \cdot \left(t + \frac{sin(2t)}{2}\right)$$

$$\frac{a^2}{2}\left(t + \sin(t)\cos(t)\right)$$

$$t = \arcsin(\frac{x}{a}) \qquad \cos(x) = \sqrt{1 - \sin^2(x)}$$

$$\frac{a^2}{2}\left(\arcsin(\frac{x}{a}) + \sin(\arcsin(\frac{x}{a}))\cos(\arcsin(\frac{x}{a}))\right)$$

$$\frac{a^2 \cdot \arcsin(\frac{x}{a})}{2} + \frac{x}{a} \cdot \sqrt{1 - sin^2(\arcsin(\frac{x}{a}))}$$

$$\frac{a^2 \cdot \arcsin(\frac{x}{a})}{2} + \frac{x}{a} \cdot \sqrt{1 - (\frac{x}{a})^2}$$

$$\left[\frac{a^2 \cdot \arcsin(\frac{x}{a})}{2} + \frac{x}{a} \cdot \sqrt{1 - (\frac{x}{a})^2}\right]_0^a =$$

$$\frac{a^2 \cdot \arcsin(\frac{0}{a})}{2} + \frac{0}{a} \cdot \sqrt{1 - (\frac{0}{a})^2} - \left(\frac{a^2 \cdot \arcsin(\frac{a}{a})}{2} + \frac{a}{a} \cdot \sqrt{1 - (\frac{a}{a})^2}\right) =$$

$$\frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{a^2\pi}{4}$$